Relation between the binary Goldbach problem and binary version of Mobius sum

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Abstract .

For any real number x > 0, let $\lfloor x \rfloor$ be the largest integer not exceeding x

and $N_{\lfloor \sqrt{x} \rfloor} = \prod_{p \leq \lfloor \sqrt{x} \rfloor, p \epsilon \mathscr{P}} p$ is the product of all primes not exceeding $\lfloor \sqrt{x} \rfloor$

with $\mathscr{P}$ is the set of primes .

let 2n ⩾ 4 a positive integer and R(2n)=card{(p,q) / p+q=2n, p,q$\epsilon \mathscr{P}^2$} denotes

the number of prime couples (p,q) such that p+q=2n .

In this paper we will prove that there is two constants $C_n = \prod_{p/2n, p \neq 2} \frac{p-1}{p-2}$ and B(n)

such that R(2n)≥ $\frac{C_n}{2}(n - \sqrt{2n}) \prod_{p / \frac{N_{\lfloor \sqrt{2n} \rfloor}}{2}} (1 - \frac{2}{p})$ - B(n) for any integer n ≥ 2 .

This would help us to prove that the Goldbach conjecture is true,by making a

connection between Goldbach function R and the binary Mobius sum .

Introduction .

This article respond to a question done by the mathematician Terry Tao in Mathoverflow.net, you can find the question exactly in **https://mathoverflow.net/questions/234158/relation-between-the-binary-goldbach-problem-and-binary-version-of-mobius-sum/234177#234177 .**

In his question he ask about the relation between the Goldbach conjecture and the binary Mobius sum, in fact he was wright about his prediction, the answer can be simply obtained by making a connection between the linear Diophantine equations and sieve methods.

Respectively.

Theorem A .

The Goldbach conjecture is true .

Lemma 1.

For any real number x > 0, let $\lfloor x \rfloor$ be the largest integer

not exceeding x and $N_{\lfloor \sqrt{x} \rfloor} = \prod_{p \leq \lfloor \sqrt{x} \rfloor, p \epsilon \mathscr{P}} p$ is the

product of all primes not exceeding $\lfloor\sqrt{x}\rfloor$, with $\mathcal{P}$ is the set of primes $\mathcal{P}=\{2,3,5,7......\}$ and let gcd(a,b) denotes the greatest common divisor of the elements (a,b)

then if $\lfloor\sqrt{x}\rfloor+1\leq n\leq x$ and $\gcd(n, N_{\lfloor\sqrt{x}\rfloor})=1 \Rightarrow$ n is a prime

Proof of Lemma 1.

Let $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}} p$

we suppose that $\gcd(n, N_{\lfloor\sqrt{n}\rfloor})=1$.

let d be a prime divisor of n $\Longrightarrow \quad 1<d\leq\lfloor\sqrt{x}\rfloor$

$\Longrightarrow \quad d/N_{\lfloor\sqrt{x}\rfloor}$

$\Longrightarrow \quad \gcd(n, N_z)\neq 1$ Absurd

then n is a prime

Lemma 2 . (see [01])

Let $\boldsymbol{\mu}$ denotes the Mobius function

$$\text{Then} \quad \sum_{d'/\gcd(n,d)} \boldsymbol{\mu}(d') = \begin{cases} 1 & \text{if } \gcd(n,d)=1 \\ 0 & \text{if not} \end{cases}$$

Lemma 3 . (see [01])

Let f be a multiplicative function then $\sum_{d/n}$ f(d) is also multiplicative .

Lemma 4 .(see[04])

$\prod_{p\leqslant x, p\neq 2}\left(1-\frac{2}{p}\right)\sim\frac{1}{\log(x)^2}$, for all sufficiently large x

Lemma 5 .(see [05])

Let a,b and c , any given integers and let ax+by=c be a diophantine equation, then ax+by=c has a solution iff gcd(a,b)/c .

And if (x_0,y_0) is a particular solution of ax+by=c

then there exists an integer k such that $(x_0+\frac{kb}{\gcd(a,b)}, y_0-\frac{ka}{\gcd(a,b)})$ is the set of solutions .

Lemma 6.

Let $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}} p$ and $d_1/N_{\lfloor\sqrt{x}\rfloor}$.

then $d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1\wedge d_2=1 \iff d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$,

Proof of Lemma 6.

Let $N_{\lfloor\sqrt{x}\rfloor}=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}} p$ and $d_1/N_{\lfloor\sqrt{x}\rfloor}$.

1- we suppose that $d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$.

we have $d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1} \Rightarrow d_2d_1/N_{\lfloor\sqrt{x}\rfloor} \Rightarrow d_2/N_{\lfloor\sqrt{x}\rfloor}$

and since $N_{\lfloor\sqrt{x}\rfloor}$ is squarefree and $d_1d_2/N_{\lfloor\sqrt{x}\rfloor}$ then $d_1\wedge d_2=1$

this means that $d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1} \Rightarrow d_2/N_{\lfloor\sqrt{x}\rfloor}$ and $d_1\wedge d_2=1$

2- we suppose that $d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1\wedge d_2=1$.

we have $d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1/N_{\lfloor\sqrt{x}\rfloor}$ and $d_1\wedge d_2=1 \Rightarrow d_2d_1/N_{\lfloor\sqrt{x}\rfloor}$

$$\Rightarrow d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$$

then from 1 and 2 we obtain the equivalence .

$d_2/N_{\lfloor\sqrt{x}\rfloor}, d_1\wedge d_2=1 \iff d_2/\frac{N_{\lfloor\sqrt{x}\rfloor}}{d_1}$

Lemma 7 . (see [06])

Let $\tau(n)$ denotes the number of divisors of n .

for all $\varepsilon > 0$, $\tau(n)=o(n^{\varepsilon})$

Proof of Theorem A .

Let $n \geq 2$, $\mathcal{P}$ denotes the set of primes R(2n)=card{(p,q) / p+q=2n p,q$\epsilon\mathcal{P}^2$}

denotes the number of couple of primes (p,q) such that p+q=2n.

For any real number x > 0, let $\lfloor x\rfloor$ be the largest integer not exceeding x.

Let R'(2n)=card{ (p,q)/p+q=2n , $\lfloor\sqrt{2n}\rfloor<p\leq n$, $n\leq q<2n-\lfloor\sqrt{2n}\rfloor$, p,q$\epsilon\mathcal{P}^2$} denotes the number of couple of primes,(p,q) such that $\lfloor\sqrt{2n}\rfloor<p\leq n$, $n\leq q<2n-\lfloor\sqrt{2n}\rfloor$ and p+q=2n and R"(2n)=card{(p,q)/p+q=2n, $1<p\leq\lfloor\sqrt{2n}\rfloor$, $2n-\lfloor\sqrt{2n}\rfloor\leq q<2n$,p,q$\epsilon\mathcal{P}^2$}

from the definitions of R(2n), R'(2n) and R"(2n) we can easily prove that

R(2n)=R'(2n)+R"(2n) .

let $z=\lfloor\sqrt{2n}\rfloor$ and $N_{\lfloor\sqrt{2n}\rfloor}=N_z=\prod_{p\leq\lfloor\sqrt{x}\rfloor, p\epsilon\mathcal{P}} p$

By Lemma 1 we have .

R'(2n)=card{ (p,q)/p+q=2n , $\lfloor\sqrt{2n}\rfloor<p\leq n$, $n\leq q<2n-\lfloor\sqrt{2n}\rfloor$,gcd(p,$N_z$)=1,gcd(q,$N_z$)=1}

$$=\sum_{p\wedge N_z=1, z<p\leq n}\sum_{q\wedge N_z=1, n\leq q<2n-z}\sum_{p+q=2n} 1$$

$$=\sum_{p\wedge N_z=1, q\wedge N_z=1, p+q=2n, z<p\leq n} 1$$

We apply Lemma 2 on $\sum_{p\wedge N_z=1, q\wedge N_z=1, p+q=2n, z<p\leq n} 1$.

R'(2n)=$\sum_{d_1/p\wedge N_z, d_2/q\wedge N_z, p+q=2n, z<p\leq n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)$

= $\sum_{d_1/p, d_1/N_z \text{ and } d_2/q, d_2/N_z, p+q=2n, z<p\leq n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)$

=$\sum_{d_1/N_z, d_2/N_z} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{d_1/p, d_2/q, p+q=2n, z<p\leq n} 1$

But we have the equivalence .

$d_1/p, d_2/p+2n \iff \exists j, k\epsilon \mathcal{N}^{\star 2}$ such that p=jd_1 et p+2n=kd_2

Then R'(2n)=$\sum_{d_1/N_z, d_2/N_z} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{\mathrm{p=jd_1, q=kd_2, p+q=2n, z<p\leq n}} 1$

=$\sum_{d_1/N_z, d_2/N_z} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{\mathrm{jd_1+kd_2=2n, z<p=jd_1\leq n}} 1$

Problem 1 .

If we want to give an explicit formula to R'(2n) we would have to calculate the sum $\sum_{\mathrm{jd_1+kd_2=2n, z<p=jd_1\leq n}} 1$.

In fact we will find that, if gcd $(d_1, d_2)/2n$ then .

$\sum_{\mathrm{jd_1+kd_2=2n, z<p=jd_1\leq n}} 1= \frac{2n-\lfloor\sqrt{2n}\rfloor}{d_1 d_2}\mathrm{gcd}(d_1, d_2)$+O(1).

Proof of Probeme 1.

Remark 1.

We remark that the sum $\sum_{\mathrm{jd_1+kd_2=2n, z<p=jd_1\leq n}} 1$, depends only on diophantine equation $\mathrm{d_1+kd_2}$=2n , with j and k are the variables .

We set $\delta(j,k)$= { 1 if the equation $\mathrm{jd_1+kd_2}=2n$ has a solution ; 0 if not

Based on Lemma 5 we have, $\delta(j,k)$= { 1 if gcd(d_1,d_2)/2n ; 0 if not

Then , $\sum_{\mathrm{jd_1+kd_2=2n, z<p=jd_1\leq n}} 1$ =$\sum_{\mathrm{z<p=jd_1\leq n}} \delta(j,k)$

= $\sum_{\mathrm{z<p=jd_1\leq n}, \mathrm{gcd}(d_1,d_2)/2n} 1$

=$\sum_{\frac{z}{d_1}<j\leq\frac{n}{d_1}, \mathrm{gcd}(d_1,d_2)/2n, j\epsilon\mathcal{N}^*} 1$

We suppose that gcd(d_1,d_2)/2n .

By Lemma 5 , we have $\sum_{jd_1+kd_2=2n, z<p=jd_1\le n} 1 = \sum_{\frac{z}{d_1}<j\le\frac{n}{d_1}, \gcd(d_1,d_2)/2n,\ j\epsilon\mathcal{N}^*} 1$

$$= \sum_{\frac{z}{d_1}<j=j_0+\frac{td_2}{\gcd(d_1,d_2)}\le\frac{n}{d_1},\ j\epsilon\mathcal{N}^*} 1$$

$$= \sum_{\frac{z}{d_1}-j_0<\frac{td_2}{\gcd(d_1,d_2)}\leqslant\frac{n}{d_1}-j_0, t\epsilon\mathcal{Z}} 1$$

$$= \sum_{\frac{z}{d_1}-j_0<\frac{td_2}{\gcd(d_1,d_2)}\leqslant\frac{n}{d_1}-j_0\ , t\epsilon\mathcal{Z}} 1$$

$$= \sum_{\frac{\frac{z}{d_1}-j_0}{d_2}\gcd(d_1,d_2)<t\leqslant\frac{\frac{n}{d_1}-j_0}{d_2}\gcd(d_1,d_2), t\epsilon\mathcal{Z}} 1$$

$$= \lfloor\frac{\frac{n}{d_1}-j_0}{d_2}\gcd(d_1,d_2)\rfloor - \lfloor\frac{\frac{z}{d_1}-j_0}{d_2}\gcd(d_1,d_2)\rfloor+1-1$$

$$= \lfloor\frac{\frac{n}{d_1}-j_0}{d_2}\gcd(d_1,d_2)\rfloor - \lfloor\frac{\frac{z}{d_1}-j_0}{d_2}\gcd(d_1,d_2)\rfloor$$

$$= \frac{\frac{n}{d_1}-j_0}{d_2}\gcd(d_1,d_2) - \frac{\frac{z}{d_1}-j_0}{d_2}\gcd(d_1,d_2)+O(1)$$

$$= \frac{\frac{n}{d_1}-\frac{z}{d_1}}{d_2}\gcd(d_1,d_2)+O(1)$$

$$= \frac{n-z}{d_2d_1}\gcd(d_1,d_2)+O(1)$$

Then if $\gcd(d_1,d_2)/2n$,we obtain .

$\sum_{jd_1+kd_2=2n, z<p=jd_1\le n} 1 = \frac{n-\lfloor\sqrt{2n}\rfloor}{d_1d_2}\gcd(d_1,d_2)+O(1)$

Let us now return to calculate R'(2n).

By Problem 1 we have.

R'(2n) $=\sum_{d_1/N_z, d_2/N_z} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\sum_{jd_1+kd_2=2n, z<p=jd_1\le n} 1$

$$=\sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)(\frac{n-z}{d_2d_1}\gcd(d_1,d_2)+O(1))$$

$$=\sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\frac{n-z}{d_1d_2}\gcd(d_1,d_2)+ \sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)O(1)$$

$$=(n-z)\sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \frac{\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)}{d_1d_2}\gcd(d_1,d_2)+ \sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)O(1)$$

Problem 2 .

Let $\tau(n)$=$\sum_{d/n} 1$ denotes the number of divisors of n.

then the error term $\sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\mathrm{O}(1)$

is equal to $O(\tau(\mathrm{rad}(2n)))$

Proof of Problem 2 .

We set F ={d=$d_1 \wedge d_2 /\ d_1/N_z, d_2/N_z$ and $d/2n$}

Then we will obtain .

L = $\sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\mathrm{O}(1)$

$=\sum_{d\epsilon F} \sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)=d} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\mathrm{O}(1)$

$=\sum_{d\epsilon F} \sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)=d} \boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)\mathrm{O}(1)$

$=\sum_{d\epsilon F} \sum_{d_1/N_z} \boldsymbol{\mu}(d_1)\sum_{d_2/N_z, \gcd(d_1,d_2)=d} \boldsymbol{\mu}(d_2)\mathrm{O}(1)$

$=\sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2)\mathrm{O}(1)$

Let $|x|$ denotes the absolute value of x ,then we have.

$$\sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2) \leq \left|\sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d\epsilon F} \left|\sum_{\frac{d_1}{d}/\frac{N_z}{d}} \boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} |\boldsymbol{\mu}(d_1)|\left|\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2)\right|$$

$$\leq \sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \left|\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2)\right|$$

By Lemma 6 , we have $\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2) = \sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)$.

Then $\sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d}, \gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1} \boldsymbol{\mu}(d_2) \leq \sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \left|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}(d_2)\right|$

$$\leq \sum_{d\epsilon F} \sum_{\frac{d_1}{d}/\frac{N_z}{d}} \left|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}} \boldsymbol{\mu}\left(\frac{d_2}{d}d\right)\right|$$

Since d_2 is a squarefree then $\gcd(\frac{d_2}{d}, d)=1$.

then $\boldsymbol{\mu}\left(\frac{d_2}{d}d\right)= \boldsymbol{\mu}(d)\boldsymbol{\mu}\left(\frac{d_2}{d}\right)$.

Then $\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\boldsymbol{\mu}(d_2) \leq \sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}\left(\frac{d_2}{d}\right)\boldsymbol{\mu}(d)|$

$$\leq \sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}\left(\frac{d_2}{d}\right)|$$

By Lemma 2 we have $\sum_{\frac{d_1}{d}/\frac{N_z}{d}}|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}\left(\frac{d_2}{d}\right)| =|\boldsymbol{\mu}\left(\frac{N_z}{d}\right)|$

Let $d\epsilon F$,then by the definition of F $=\{d=d_1 \wedge d_2/,\ d_1/N_z, d_2/N_z, d/2n\}$,

d will be a squarefree .

then $\frac{N_z}{d}$ is also a squarefree

Then $\sum_{\frac{d_1}{d}/\frac{N_z}{d}}|\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\boldsymbol{\mu}(d_2)| =|\boldsymbol{\mu}\left(\frac{N_z}{d}\right)|=1$

Then $\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\boldsymbol{\mu}(d_2) \leq \sum_{d\epsilon F} 1$

Remark 2.

since N_z is squarefree then F $=\{d=d_1 \wedge d_2/,\ d_1/N_z, d_2/N_z, d/\mathrm{rad}(2n)\}$

$=\{d/\mathrm{rad}(2n)\ /\ d\leqslant \lfloor\sqrt{2n}\rfloor\}$

We have $\tau(\mathrm{rad}(2n))$=card{ d/rad(2n)}

=card{{ d/rad(2n)/ d$\leqslant \lfloor\sqrt{2n}\rfloor$} $\cup$ {d/rad(2n) / d$\geqslant \lfloor\sqrt{x}\rfloor$}}

=card {F $\cup${d/rad(2n) / d$\geqslant \lfloor\sqrt{x}\rfloor$}}

This means that, F $\subset \{d/\mathrm{rad}(2n)\ \}$

Then we can deduce that, rad(F) $\leq \tau(\mathrm{rad}(2n))$

We have $\sum_{d\epsilon F} 1$=rad(F) ,

By Remark 2 , we have rad(F) $\leq \tau(\mathrm{rad}(2n))$

Then , $\sum_{d\epsilon F} 1 \leq \tau(\mathrm{rad}(2n))$

Then ,$\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\boldsymbol{\mu}(d_1)\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\boldsymbol{\mu}(d_2) \leq \tau(\mathrm{rad}(2n))$

Which means that $\sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)O(1)=O(\tau(\mathrm{rad}(2n)))$

Result 1.

The error term of R'(2n) is equal to O($\tau(\mathrm{rad}(2n))$) , this is the most important result in this paper ,because in the coming sections we will prove that the main part of R'(2n) is much more greater than the error term O($\tau(\mathrm{rad}(2n))$)

Let us return to calculate R'(2n) .

$$R'(2n)=(n-z)\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\frac{\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)}{d_1d_2}\gcd(d_1,d_2)+\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)O(1)$$

By the Problem 2 we have .

$$R'(2n)=(n-z)\sum_{d_1/N_z,d_2/N_z,\gcd(d_1,d_2)/2n}\frac{\boldsymbol{\mu}(d_1)\boldsymbol{\mu}(d_2)}{d_1d_2}\gcd(d_1,d_2)+O(\tau(\mathrm{rad}(2n)))$$
$$=(n-z)\sum_{d_1/N_z}\frac{\boldsymbol{\mu}(d_1)}{d_1}\sum_{d_2/N_z,\gcd(d_1,d_2)/2n}\frac{\boldsymbol{\mu}(d_2)}{d_2}\gcd(d_1,d_2)+O(\tau(\mathrm{rad}(2n)))$$
$$=(n-z)\sum_{d\epsilon F}\sum_{d_1/N_z}\frac{\boldsymbol{\mu}(d_1)}{d_1}\sum_{d_2/N_z,\gcd(d_1,d_2)=d}\frac{\boldsymbol{\mu}(d_2)}{d_2}d+O(\tau(\mathrm{rad}(2n)))$$
$$=(n-z)\sum_{d\epsilon F}\sum_{d_1/N_z}\frac{\boldsymbol{\mu}(d_1)}{d_1}\sum_{d_2/N_z,\gcd(d_1,d_2)=d}\frac{\boldsymbol{\mu}(d_2)}{d_2}d+O(\tau(\mathrm{rad}(2n)))$$
$$=(n-z)\sum_{d\epsilon F}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}d\right)}{d_1}\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}d\right)}{d_2}d+O(\tau(\mathrm{rad}(2n)))$$
$$=(n-z)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}d\right)}{\frac{d_1}{d}}\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}d\right)}{\frac{d_2}{d}}+O(\tau(\mathrm{rad}(2n)))$$

Since $\gcd(\frac{d_1}{d},d)=1$ and $\gcd(\frac{d_2}{d},d)=1$,

Then $\boldsymbol{\mu}\left(\frac{d_1}{d}d\right)=\boldsymbol{\mu}(d)\boldsymbol{\mu}\left(\frac{d_1}{d}\right)$ and $\boldsymbol{\mu}\left(\frac{d_2}{d}d\right)=\boldsymbol{\mu}(d)\boldsymbol{\mu}\left(\frac{d_2}{d}\right)$.

We obtain.

$$R'(2n)=(n-z)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}\boldsymbol{\mu}(d)^2+O(\tau(\mathrm{rad}(2n)))$$

Since d is a squarefree then $\boldsymbol{\mu}(d)^2=1$, Then

$$R'(2n)=(n-z)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\sum_{\frac{d_2}{d}/\frac{N_z}{d},\gcd\left(\frac{d_1}{d},\frac{d_2}{d}\right)=1}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}+O(\tau(\mathrm{rad}(2n)))$$

By Lemma 6 we have .

$$R'(2n)=(n-z)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}+O(\tau(\mathrm{rad}(2n)))$$

Since $\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}$ is multiplicative then by Lemma 3 ,$\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}$ is also multiplicative

and $$\sum_{\frac{d_2}{d}/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}\frac{\boldsymbol{\mu}\left(\frac{d_2}{d}\right)}{\frac{d_2}{d}}=\prod_{p/\frac{\frac{N_z}{d}}{\frac{d_1}{d}}}(1-\frac{1}{p})$$
$$=\frac{\prod_{p/\frac{N_z}{d}}(1-\frac{1}{p})}{\prod_{p/\frac{d_1}{d}}(1-\frac{1}{p})}$$

$$R'(2n) = (n\text{-}z)\sum_{d\epsilon F}\frac{1}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}}\frac{\prod_{p/\frac{N_z}{d}}(1-\frac{1}{p})}{\prod_{p/\frac{d_1}{d}}(1-\frac{1}{p})}+O(\tau(\mathrm{rad}(2n)))$$

$$= (n\text{-}z)\sum_{d\epsilon F}\frac{\prod_{p/\frac{N_z}{d}}(1-\frac{1}{p})}{d}\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}\prod_{p/\frac{d_1}{d}}(1-\frac{1}{p})}+O(\tau(\mathrm{rad}(2n)))$$

We apply again Lemma 3 on $\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}\prod_{p/\frac{d_1}{d}}(1-\frac{1}{p})}$ we obtain .

$$\sum_{\frac{d_1}{d}/\frac{N_z}{d}}\frac{\boldsymbol{\mu}\left(\frac{d_1}{d}\right)}{\frac{d_1}{d}\prod_{p/\frac{d_1}{d}}(1-\frac{1}{p})} = \prod_{p/\frac{N_z}{d}}\left(1-\frac{1}{p\left(1-\frac{1}{p}\right)}\right)$$

$$= \prod_{p/\frac{N_z}{d}}\left(1-\frac{1}{p-1}\right)$$

$$= \prod_{p/\frac{N_z}{d}}\left(\frac{p-2}{p-1}\right)$$

Then $R'(2n) = (n\text{-}z)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}(1-\frac{1}{p})\prod_{p/\frac{N_z}{d}}(\frac{p-2}{p-1}) + O(\tau(\mathrm{rad}(2n)))$

$$= (n\text{-}z)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(\frac{p-1}{p}\right)\left(\frac{p-2}{p-1}\right) + O(\tau(\mathrm{rad}(2n)))$$

$$= (n\text{-}z)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(\frac{p-2}{p}\right) + O(\tau(\mathrm{rad}(2n)))$$

$$= (n\text{-}z)\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right) + O(\tau(\mathrm{rad}(2n)))$$

We have $F = \{d = d_1 \wedge d_2/,\ d_1/N_z, d_2/N_z, d/\mathrm{rad}(2n)\}$

$= \{d/\mathrm{rad}(2n)\ /\ d \leqslant \lfloor\sqrt{x}\rfloor\}$

Then $F \subset \{d/\mathrm{rad}(2n)\}$.

which means that.

$$\sum_{d\epsilon F}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right) \geq \sum_{d/\mathrm{rad}(2n)}\frac{1}{d}\prod_{p/\frac{N_z}{d}}\left(1-\frac{2}{p}\right)$$

We have $\sum_{d/\mathrm{rad}(2n)}\frac{1}{d}\prod_{p/\frac{N_z}{d}}(1-\frac{2}{p}) = \prod_{p/\frac{N_z}{2}}(1-\frac{2}{p})\sum_{d/\mathrm{rad}(2n)}\frac{\prod_{p/\frac{N_z}{d}}(1-\frac{2}{p})}{d\prod_{p/\frac{N_z}{2}}(1-\frac{2}{p})}$

$$= \prod_{p/\frac{N_z}{2}}\left(1-\frac{2}{p}\right)\sum_{d/\mathrm{rad}(2n)}\frac{\prod_{p/\frac{N_z}{d}}(1-\frac{2}{p})}{d\prod_{p/\frac{N_z}{2}}(1-\frac{2}{p})}$$

If gcd(d,2)≠2 then $\frac{N_z}{d}$ is even and $2/\frac{N_z}{d}$

Which means that $\prod_{p/\frac{N_z}{d}}(1-\frac{2}{p}) = 0$.

Then $\sum_{d/\mathrm{rad}(2n)} \frac{1}{d}\prod_{p/\frac{N_z}{d}} (1-\frac{2}{p}) = \prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{d/\mathrm{rad}(2n),\gcd(d,2)=2} \frac{\prod_{p/\frac{N_z}{d}} (1-\frac{2}{p})}{\mathrm{d}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})}$

$$=\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{\prod_{p/\frac{N_z}{2\frac{d}{2}}} (1-\frac{2}{p})}{\mathrm{d}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})}$$

$$=\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{\prod_{p/\frac{N_z}{2\frac{d}{2}}} (1-\frac{2}{p})}{\mathrm{d}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})}$$

$$=\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})}{\mathrm{d}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\prod_{p/\frac{d}{2}} (1-\frac{2}{p})}$$

$$=\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{1}{\mathrm{d}\prod_{p/\frac{d}{2}} (1-\frac{2}{p})}$$

$$=\frac{1}{2}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{2}{\mathrm{d}\prod_{p/\frac{d}{2}} (1-\frac{2}{p})}$$

$$=\frac{1}{2}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{1}{\frac{\mathrm{d}}{2}\prod_{p/\frac{d}{2}} (1-\frac{2}{p})}$$

Since $\frac{1}{\frac{\mathrm{d}}{2}\prod_{p/\frac{d}{2}} (1-\frac{2}{p})}$ is multiplicative then by Lemma 3 we have

$\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{1}{\frac{\mathrm{d}}{2}\prod_{p/\frac{d}{2}} (1-\frac{2}{p})}$ also multiplicative

And $\sum_{\frac{d}{2}/\frac{\mathrm{rad}(2n)}{2}} \frac{1}{\frac{\mathrm{d}}{2}\prod_{p/\frac{d}{2}} (1-\frac{2}{p})} = \prod_{p/\frac{\mathrm{rad}(2n)}{2}} (1+\frac{1}{p\left(1-\frac{2}{p}\right)})$

$$=\prod_{p/\frac{\mathrm{rad}(2n)}{2}} (1+\frac{1}{p-2})$$

$$=\prod_{p/\frac{\mathrm{rad}(2n)}{2}} \frac{p-1}{p-2}$$

$$= \prod_{p/2n,p\neq 2} \frac{p-1}{p-2}$$

Then $\sum_{d/\mathrm{rad}(2n)} \frac{1}{d}\prod_{p/\frac{N_z}{d}} (1-\frac{2}{p}) = \frac{\prod_{p/2n,p\neq 2} \frac{p-1}{p-2}}{2}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})$

We set $\mathrm{C}_n=\prod_{p/2n,p\neq 2} \frac{p-1}{p-2}$

Then $\sum_{d/\mathrm{rad}(2n)} \frac{1}{d}\prod_{p/\frac{N_z}{d}} (1-\frac{2}{p})=\frac{\mathrm{C}_n}{2}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})$

Then $,\sum_{d\epsilon F} \frac{1}{d}\prod_{p/\frac{N_z}{d}} (1-\frac{2}{p}) \geq \frac{\mathrm{C}_n}{2}\prod_{p/\frac{N_z}{2}} (1-\frac{2}{p})$

Which means that R'(2n) $\geqslant \frac{C_n}{2}(n-z)\prod_{p/\frac{Nz}{2}}(1-\frac{2}{p})$ - $\tau(\mathrm{rad}(2n))$

such that $C_n=\prod_{p/2n,p\neq 2}\frac{p-1}{p-2}$.

We know that , R(2n) $\geqslant$R'(2n) .

Then R(2n) $\geqslant \frac{C_n}{2}(n-\lfloor\sqrt{2n}\rfloor)\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})$ - B(n)

such that B(n)=$\tau(\mathrm{rad}(2n))$ and $C_n=\prod_{p/2n,p\neq 2}\frac{p-1}{p-2}$.

By Lemma 4 we have $\prod_{p\leqslant x,p\neq 2}\left(1-\frac{2}{p}\right)\sim\frac{1}{\log(x)^2}$, for all sufficiently large x .

By Lemma 7 we have for $\varepsilon=\frac{1}{2}$, B(n) = $\tau(\mathrm{rad}(2n))$

$$= o((\mathrm{rad}(2n))^{\frac{1}{2}})$$

$$=o((2n)^{\frac{1}{2}})$$

Then .

$$\frac{C_n}{2}(n-\lfloor\sqrt{2n}\rfloor)\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p}) - B(n) = \frac{C_n}{2}(n-\lfloor\sqrt{2n}\rfloor)\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p}) - o((2n)^{\frac{1}{2}})$$

$$=\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})(1-\frac{1}{\sqrt{2n}}-o(\frac{(2n)^{\frac{1}{2}}}{\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})}))$$

We have $\frac{(2n)^{\frac{1}{2}}}{\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})} = \frac{\sqrt{2}}{\frac{C_n}{2}\sqrt{n}\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})}$

$$\sim\frac{\sqrt{2}\log(\sqrt{2n})^2}{\frac{C_n}{2}\sqrt{n}}$$

$$\sim\frac{\sqrt{2}\log(2n)^2}{2C_n\sqrt{n}}$$

Since $\frac{\sqrt{2}\log(2n)^2}{2C_n\sqrt{n}}\rightarrow 0$ when x$\rightarrow\infty$, then $\frac{(2n)^{\frac{1}{2}}}{\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})}=o(1)$

Then $1-\frac{1}{\sqrt{2n}}-o(\frac{(2n)^{\frac{1}{2}}}{\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})}=1+o(1)$

Then $\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})(1-\frac{1}{\sqrt{2n}}-o(\frac{(2n)^{\frac{1}{2}}}{\frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})})) = \frac{C_n}{2}n\prod_{p\leq\lfloor\sqrt{2n}\rfloor,p\neq 2}(1-\frac{2}{p})(1+o(1))$

Then $\frac{C_n}{2}n\prod_{p\le\lfloor\sqrt{2n}\rfloor, p\neq 2}(1-\frac{2}{p})(1-\frac{1}{\sqrt{2n}}-o(\frac{(2n)^{\frac{1}{2}}}{\frac{C_n}{2}n\prod_{p\le\lfloor\sqrt{2n}\rfloor, p\neq 2}(1-\frac{2}{p})}))\sim \frac{C_n}{2}n\prod_{p\le\lfloor\sqrt{2n}\rfloor, p\neq 2}(1-\frac{2}{p})$

$$\sim\frac{C_n}{2}\frac{4n}{\log(2n)^2}$$

$$\sim 2C_n\frac{n}{\log(2n)^2}$$

This means that R(2n) $\to\infty$ when n $\to\infty$

This confirm that The Goldbach conjecture is true

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